Linear Temporal Logic LTL: Basis for Admissible Rules

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Abstract

The object of our study is the propositional linear temporal logic LTL with operations Until and Next. We deal with inference rules admissible in LTL. Earlier the decidability of LTL w.r.t. to admissibility was shown. That, in particular, allowed to suggest a recursive infinite basis for admissible rules. However, that result gives no information about the structure and properties of such basis, thereby not allowing to use it for inferences. The current article aims to cover this gap and to provide an explicit (infinite) basis for rules admissible in LTL.

Keywords: Temporal logic, admissible rules, bases for inference rules.

1 Introduction

The concept of admissibility for inference rule was introduced into consideration by P. Lorenzen in 1955. In a logic \( L \), a rule of inference is admissible if the set of theorems of \( L \) is closed with respect to applications of this rule. Admissible rules constitute the most general class of rules, which are, in a sense, compatible with the logic. In particular, this class is the biggest collection of rules which can be safely added to an axiomatic system of the logic. Research into the topic of admissible rules was stimulated by a question of Friedman [6], whether admissibility of rules in the intuitionistic logic IPC is decidable. This problem, as well as its analogues for various transitive modal logics (e.g. \( S4 \), \( K4 \), \( SS \), \( S4.3 \)) were studied and solved affirmatively by Rybakov (cf. [40], see [37, 39] for first solutions for IPC and \( S4 \)) by finding decision algorithms for these problems. Later, a solution of admissibility problem in IPC based on proof-theoretic technique was found by Roziere [36]. Using another independent approach, Ghilardi found a solution of admissibility problem for IPC and \( S4 \) via unification and projective formulas (cf. [12]). This technique and its enhancements were also successfully applied to unification and projectivity problems in various other logics (cf. [11, 13–15]).

Kuznetsov’s problem, whether \( IPC \) (and \( S4 \)) has a finite basis for admissible rules was answered in negative also by Rybakov (cf. [38] and [40], respectively). However, it was not clear, which explicit bases may describe rules admissible in IPC. This question was answered by Iemhoff [17], where it was shown that de Jongh-Visser rules indeed form a basis for rules admissible in IPC. A modal variation of this basis forms a basis for rules admissible in \( S4 \) [41]. Starting from [17], the structure of possible admissible rules in terms related to proof theory has been intensively studied [1, 18–20, 33, 34].
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Explicit bases were found, but the question whether the independent bases are possible, was open. It was solved by Jerábek in [23], who found an independent basis for rules admissible in IPC. In the paper [21], he also presents bases for admissible rules in several modal logics related to S4. In the paper [22], the same author gives explicitly strict complexity estimates for admissibility decision algorithms.

All these results concerning admissibility were impressive, but yet it had been also visible that they were primarily focused on non-classical logics related to IPC and transitive modal logics (like S4 and K4). Therefore, a move to logics with another background, or important from the viewpoint of applications, was deemed desirable for the development of the area. In this line, recently, the decidability problem for linear temporal logic LT was solved by Rybakov in [44]. Also the cases of temporal logics based on integers Z [42], and temporal logics admitting universal modality [43] were studied and admissibility decision algorithms were found. Recently, the research was done in Lukasiewicz logic and algorithms solving decidability and bases for admissible rules were presented [24, 25].

As was mentioned above, the decidability problem for admissibility in the linear temporal logic LT was solved in [44], but the question about finding explicit bases for rules admissible in LT was open. Therefore, we dedicate our article to solution of this problem.

The logic LT is a well-known important non-classical logic with the background in artificial intelligence (AI) and computer science (CS). LT is a temporal logic with operations Until and Next referring to (linear) time. LT was widely used in AI and CS research (cf. e.g. [4, 31, 32]) and linear temporal logics were used by mathematicians and philosophers for reasoning about knowledge and time (cf. e.g. [9, 46]). LT, as well as various other temporal logics, have numerous applications to problems arising in the theory of computation (cf. [3]). For checking satisfiability in LT a subtle technique based on automata theory was developed (cf. [5, 7, 47, 48]). The decidability of LT also can be shown by bounded finite model property (cf. [45]).

To the best of our knowledge, first axiomatization for LT was suggested in [8] (cf. also [27]). Later M. Lange in [29] also gave a decision procedure and a complete axiomatization for LT extended by Past. Various axiomatic systems for LT and related logics were suggested and discussed in [28]. We would like to extend these cornerstone results towards finding axiomatization for rules admissible in LT. The main result of our article is the solution of this problem: we construct an explicit basis for inference rules admissible in LT.

2 Definitions and notation

For reader reference, we start from definitions, notation and basic tools necessary for this article. To define formulas of LT, we fix an enumerable set Var=x_1,x_2,x_3,... of propositional variables. The formulas over the propositional language

\[
\mathcal{L} := \{\lor^2, \land^2, \rightarrow^2, \neg^1, N^1, U^2\}
\]

are defined by the following grammar:

\[
\alpha ::= x_i | \alpha_1 \land \alpha_2 | \alpha_1 \lor \alpha_2 | \alpha_1 \rightarrow \alpha_2 | \neg \alpha_1 | N\alpha_1 | \alpha_1 U \alpha_2
\]

The set of all \(\mathcal{L}\)-formulas is denoted by \(Fm_{\mathcal{L}}\). For a formula \(\alpha\), \(Var(\alpha)\) will denote the set of all variables occurring in \(\alpha\). We will also use the common abbreviations: \(\top := x \lor \neg x\), \(\bot := x \land \neg x\), \(\lozenge \alpha := \top U \alpha\), \(\Box \alpha := \neg \lozenge \neg \alpha\).
Let \( \mathcal{N} := (\mathbb{N}, \mathbb{N}) \) be the Kripke structure based on natural numbers \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \), where \( \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N} \) (\( N \) stands for Next) is defined as \( \mathbb{N} := \{(i, i+1) | i \in \mathbb{N}\} \). Logic \( \text{LTL} \) (Linear Temporal Logic) is defined to be the set of all \( L \)-formulas valid in all Kripke models based on the frame \( \mathcal{N} \), i.e. \( \text{LTL} \) is the set of all formulas \( \alpha \in \text{Fm}_L \) such that for any valuation of variables \( \nu: \text{Var(} \alpha \text{)} \rightarrow 2^\mathbb{N} \), \( \mathcal{N} \models_\nu \alpha \), where the truth-values of non-Boolean operations \( N \) and \( U \) are defined as follows:

\[
(\mathcal{N}, i) \models_\nu N \alpha \iff (\mathcal{N}, i+1) \models_\nu \alpha,
\]

\[
(\mathcal{N}, i) \models_\nu U \beta \iff \exists j \geq i [(\mathcal{N}, j) \models_\nu \beta \land \forall k (i \leq k < j \implies (\mathcal{N}, k) \models_\nu \alpha)].
\]

Note that we will be using \( N \) both as an operation symbol, as well as the name for the corresponding accessibility relation. Also note that in some sources (e.g. [28]) the above defined operator \( U \) is referred to as weak-\text{until}.

By \( M_1 \sqcup M_2 \) we denote the disjoint union of models \( M_1 \) and \( M_2 \).

**Definition 1**

For a fixed \( n \), let

\[
\mathbf{N} := \bigsqcup_{\nu} \mathbf{N}_\nu,
\]

where \( \mathbf{N}_\nu := \langle \mathcal{N}_\nu, \nu \rangle \) are taken for all valuations \( \nu: \{x_1, \ldots, x_n\} \rightarrow 2^\mathbb{N} \).

We will denote by \( \mathbf{N}[i, \infty] \) the open submodel of \( \mathbf{N}_\nu \), the elements of which start from \( i \in \mathbf{N}_\nu \). Obviously, \( \mathbf{N}[i, \infty] \) is isomorphic to some \( \mathbf{N}_\mu \). It follows directly from the definition of \( \text{LTL} \) that \( \mathbf{N} \) is an \( n \)-characterizing model for \( \text{LTL} \), i.e. for every \( L \)-formula \( \alpha(x_1, \ldots, x_n) \):

\[
\alpha \in \text{LTL} \iff \mathbf{N} \models_\nu \alpha.
\]

The following formulas are theorems of \( \text{LTL} \):

- \( Nx \rightarrow \Box x \) \hspace{1cm} (T1)
- \( \Box(x \rightarrow y) \lor \Box(y \rightarrow x) \) \hspace{1cm} (T2)
- \( \Box x \rightarrow (x \lor N \Box x) \) \hspace{1cm} (T3)
- \( (x \land \Box(x \rightarrow N x)) \rightarrow \Box x \) \hspace{1cm} (T4)
- \( x U y \leftrightarrow \Box y \land (y \lor (x \land N(x U y))) \) \hspace{1cm} (T5)

that can be verified by direct check of their validity in \( \mathcal{N} \) (cf. [28]).

A rule or, synonymously, an inference rule \( r \) is an expression

\[
r := \psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n) \rightarrow \psi(x_1, \ldots, x_n),
\]

where \( \psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n) \) and \( \psi(x_1, \ldots, x_n) \) are arbitrary formulas constructed from variable letters, as explained above. The formula \( \psi(x_1, \ldots, x_n) \) is the conclusion of \( r \), formulas \( \psi_j(x_1, \ldots, x_n) \) are the premises of \( r \), letters \( x_i \) are variables of \( r \).
First, we define valid rules. Let $\mathcal{F}$ be a frame, e.g. our linear temporal frame $\mathcal{N}$, with a valuation $V$ of all variables from a rule $r := \psi_1, \ldots, \psi_n / \psi$. The rule $r$ is said to be valid in the Kripke structure $\langle \mathcal{F}, V \rangle$ (notation: $\mathcal{F} \models r$, or $\mathcal{F} \models_{V} r$) if $(\langle \mathcal{F}, V \rangle \models \bigwedge_{1 \leq i \leq m} \psi_i ) \implies (\langle \mathcal{F}, V \rangle \models \psi)$. That is, $r$ is valid in $\langle \mathcal{F}, V \rangle$ if the following holds. If all premises of $r$ are valid in $\mathcal{F}$ w.r.t. $V$ (which means they are true at all worlds from $\mathcal{F}$), then the conclusion is valid in $\mathcal{F}$ w.r.t. $V$ (again in all worlds from $\mathcal{F}$) as well.

A rule $r$ is valid in a frame $\mathcal{F}$ (notation: $\mathcal{F} \equiv r$) if, for any valuation $V$, $\mathcal{F} \equiv_{V} r$. If a rule $r$ is not valid in $\mathcal{F}$ w.r.t. some valuation $V$, then we say $r$ is refuted in $\mathcal{F}$, or refuted in $\mathcal{F}$ by $V$, and write $\mathcal{F} \equiv_{V} r$. Note that this definition of valid rules is equivalent to the notion of valid modal sequents from [26], where a theory of sequent-axiomatic classes is developed. Also the notion of valid rules can be reduced to validity of formulas in the extension of the language with universal modality (cf. [10]). Based on these results, some relevant approach to validity of rules can be derived. Some examples of rules valid in $\mathcal{LT}_{L}$, i.e. valid in the frame $\mathcal{N}$ defining the logic $\mathcal{LT}_{L}$ are: $\Box x_1 / x_1$, $x_1 / N x_1$, $\Box \Box x_1 / x_1$, $\Box \Box x_1 / \Box \Box x_1$ and $\Diamond \Diamond x_1 / \Box \Box x_1$, where $\Diamond \alpha := T U \alpha$, $\Box \alpha := \neg \alpha \land \alpha$ as above (cf. [44]). It is easy to accept that valid rules correctly describe logical consequence. But a reasonable question is whether only valid rules may be applied to study logical inference. The strongest class of possible structural logical rules correct for a given logic $\Lambda$, the main object of our article—admissible rules, was introduced into consideration by Lorenzen [30]. Given an arbitrary propositional logic $\Lambda$, $\text{Fm}_{\Lambda}$ is the set of all formulas in the language of $\Lambda$. Let $r := \psi(x_1, \ldots, x_n), \ldots, \phi_m(x_1, \ldots, x_n) / \psi(x_1, \ldots, x_n)$ be a rule in the language of $\Lambda$.

**Definition 2**

Rule $r$ is admissible in a logic $\Lambda$ if for all $\alpha_1, \ldots, \alpha_n \in \text{Fm}_{\Lambda},$

$$[\bigwedge_{1 \leq i \leq m} \psi_i (\alpha_1, \ldots, \alpha_n \in \Lambda)] \implies [\psi(\alpha_1, \ldots, \alpha_n \in \Lambda)].$$

For any logic $\Lambda$, any rule valid in all $\Lambda$-frames or derivable in an axiomatic system for $\Lambda$ must be admissible for $\Lambda$ (very well-known fact), the converse is not always true. Examples of invalid and non-derivable rules, which are anyway admissible (for the intuitionistic logic and some modal logics) can be found in, e.g. [16, 35, 40].

Some simple examples of rules that are admissible but not valid in the frame $\mathcal{N}$, which generates the logic $\mathcal{LT}_{L}$ (consequently, these rules are not derivable in any axiomatic system for $\mathcal{LT}_{L}$) were presented in [44], e.g. $\text{N} x_1 / x_1, \text{N} x_1 \rightarrow \text{N} x_2 / x_1 \rightarrow x_2, \text{N} x_1 \text{U} x_2 / x_1 \text{U} x_2$ are admissible but invalid in $\mathcal{N}$. In general, for every formula $\phi(x_1, \ldots, x_n)$ the rule $\phi(\text{N} x_1, \ldots, \text{N} x_n) / \phi(x_1, \ldots, x_n)$ is admissible in $\mathcal{LT}_{L}$.

### 3 Stone spaces for $\mathcal{LT}_{L}$

In this section, we will describe the basics of algebraic semantics and a rudimentary theory of Stone spaces for $\mathcal{LT}_{L}$. An $\mathcal{LT}_{L}$-algebra is an algebra

$$\mathfrak{A} = \langle A, \lor, \land, \neg, N, U \rangle,$$

where

(1) $\langle A, \lor, \land, \neg, \bot, \top \rangle$ is a Boolean algebra;
(2) $N : A \rightarrow A$, $U : A \times A \rightarrow A$; and
(3) $\mathfrak{A} \models \alpha = \beta$ for every bi-implication $\alpha \leftrightarrow \beta \in \mathcal{LTL}$.

Instead of $\mathfrak{A} \models \alpha = \top$, we will customary write $\mathfrak{A} \models \alpha$.

**Definition 3**

Given an $n$-generated $\mathcal{LTL}$-algebra $\mathfrak{A} = \mathfrak{A}(a_1, \ldots, a_n)$ with the fixed generators $a_1, \ldots, a_n$, the Kripke model

$$\mathfrak{A}^* := (U(\mathfrak{A}), N, R, v)$$

has the following structure:

1. $U(\mathfrak{A})$ is the set of ultrafilters of the Boolean algebra $\langle A, \lor, \land, \neg, \bot, \top \rangle$.
2. For all $u, v \in U(\mathfrak{A})$
   $$uNv \iff (\forall a \in A)(Na \in u \iff a \in v).$$
   $$uRv \iff (\forall a \in A)(a \in v \Rightarrow \Diamond a \in u).$$
3. $v$ is the valuation of variables $x_1, \ldots, x_n$ defined as follows:
   $$u \in v(x_i) \iff a_i \in u.$$

Note that the definition of the valuation in $\mathfrak{A}(\bar{a})^*$ depends on the choice of generators. We will also customary write $u \in \mathfrak{A}^*$ instead of $u \in U(\mathfrak{A})$.

**Definition 4**

Suppose $\mathfrak{A} = \mathfrak{A}(\bar{a})$ is a $\mathcal{LTL}$-algebra. For the Kripke model $\mathfrak{A}^*$, we define the forcing relation $\models_{\mathfrak{A}^*} \subseteq U(\mathfrak{A}) \times \text{Fm}_L[\bar{x}]$ as follows:

$$u \models_{\mathfrak{A}^*} \alpha(\bar{x}) \iff \alpha(\bar{a}) \in u.$$

Further on, we will usually write $(\mathfrak{A}^*, u) \models \alpha(\bar{x})$ instead of $u \models_{\mathfrak{A}^*} \alpha(\bar{x})$.

Note that it follows from the properties of Boolean ultrafilters that truth-values determined by $\models$ agree with the usual inductive definition for the Boolean connectives.

**Lemma 1**

Suppose $\mathfrak{A} = \mathfrak{A}(\bar{a})$ is a $\mathcal{LTL}$-algebra. Then

1. $\mathfrak{A} \models \alpha(\bar{a}) \iff \mathfrak{A}^* \models \alpha(\bar{x})$ and (2) for all $\alpha(\bar{x}) \in \mathcal{LTL}$: $\mathfrak{A}^* \models \alpha(\bar{x})$.

**Proof.** (1) $(\Rightarrow)$ follows directly from Definition 4, because $\alpha(\bar{a}) = \mathfrak{A} \top$ means that for all $u \in \mathfrak{A}^*$: $\alpha(\bar{a}) \in u$, therefore $(\mathfrak{A}^*, u) \models \alpha(\bar{x})$, for all $u \in \mathfrak{A}^*$, hence $\mathfrak{A}^* \models \alpha(\bar{x})$. For the other direction, suppose $\mathfrak{A} \not\models \alpha(\bar{a})$. Then $\neg \alpha(\bar{a}) \not\models \mathfrak{A} \bot$ and the set $\{ \neg \alpha(\bar{a}) \}$ can be extended to an ultrafilter $u \in \mathfrak{A}^*$, hence $(\mathfrak{A}^*, u) \not\models \alpha(\bar{x})$. (2) By definition of $\mathcal{LTL}$-algebras, $\mathfrak{A} \models \alpha(\bar{a}) = \beta(\bar{a})$, for all $\alpha(\bar{x}) \leftrightarrow \beta(\bar{x}) \in \mathcal{LTL}$. If $\alpha(\bar{x})$ is a theorem of $\mathcal{LTL}$, then $\alpha(\bar{x}) \leftrightarrow \top \in \mathcal{LTL}$, therefore $\mathfrak{A} \models \alpha(\bar{x})$. In particular, $\mathfrak{A} \models \alpha(\bar{a})$. By (1), $\mathfrak{A}^* \models \alpha(\bar{x})$, as needed.

**Lemma 2**

For each $u \in \mathfrak{A}^*$, there exists a unique $v \in \mathfrak{A}^*$ (may be the world $u$ itself), such that $uNv$.

**Proof.** Let $X := \{ a \in A \mid Na \in u \}$. Let us show that $X$ is an ultrafilter.

(1) $X$ is not empty, because $\top \in X$, since $\top = \mathfrak{A} \top = \mathfrak{A} N(a \rightarrow a) = \mathfrak{A} Na \rightarrow Na = \mathfrak{A} \top \in u$. (2) If $a, b \in X$, then $Na, Nb \in u$, therefore $Na \land Nb \in u$. Since $Na \land Nb \leftrightarrow N(a \land b) \in \mathcal{LTL}$, we have that $...$
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\[\mathfrak{A} \models Na \land Nb = Na \land b.\] Hence \(N(a \land b) \in u\), and therefore \(a \land b \in X\). (3) If \(a, a \rightarrow b \in X\), then \(Na, Na \rightarrow Nb(= \mathfrak{A} N(a \rightarrow b)) \in u\), therefore \(Nb \in u\) and \(b \in X\). (4) Suppose \(a \notin X\), then \(Na \notin u\), hence \(\neg Na = \mathfrak{A} \neg a \in u\), therefore \(\neg a \in X\). (5) Finally, if \(\perp \in X\), then \(N \perp = \mathfrak{A} N \perp = \mathfrak{A} \perp \perp \in u\), a contradiction.

By definition of \(X\), \(u \cap X\).

Let \(v \in \mathfrak{X}^*\) be an ultrafilter such that \(u \cap v\). By definition of \(N\), \(X \subseteq v\). We have already proved that \(X\) is an ultrafilter (therefore maximal), hence \(X = v\). Thus \(v\), with the property \(u \cap v\), is unique. ■

**Corollary 1**

\((\mathfrak{X}^*, u) \models N \beta(\bar{a}) \iff (\mathfrak{X}^*, N(u)) \models \beta(\bar{a})\).

Although, the truth-values of \(U\)-formulas are defined on \(\mathfrak{X}^*\) through the forcing relation \(\models\), with no direct reference to either accessibility relation \(N\) or \(R\), the operator \(\diamond\) (i.e. \(T u\)), nevertheless, in full agreement with its name, behaves like the standard possibility operator with respect to accessibility relation \(R\). This can be shown using the standard modal logic argument, which, for its importance for this article, we recount in the following lemma.

**Lemma 3**

\((\mathfrak{X}^*, u) \models \diamond \beta(\bar{a}) \iff \exists v(uRv \& (\mathfrak{X}^*, v) \models \beta(\bar{a}))\).

**Proof.** We will use that by definition of \(\models\)

\[(\mathfrak{X}^*, u) \models a(\bar{a}) \iff a(\bar{a}) \in u.\]

Recall that \(\Box := \neg \diamond \neg\). Suppose \(\diamond \beta(\bar{a}) \in u\) and let \(X := \{a \in \mathfrak{X} \mid \Box a \in u\}\). Since \(X\) is closed under finite intersections (because \(\Box (b_1 \land \ldots \land b_n) = \mathfrak{X} \Box b_1 \land \ldots \land \Box b_n\)), then all finite intersections in the set \(X \cup \{\beta(a)\}\) are non-empty. Indeed, let \(a \in X\) (i.e. \(\Box a \in u\)) and suppose \(a \land \beta(a) = \mathfrak{X} \perp\). Then

\[a \rightarrow \neg \beta(\bar{a}) = \mathfrak{X} \neg a \lor \neg \beta(\bar{a}) = \mathfrak{X} (a \land \beta(\bar{a})) = \mathfrak{X} \perp = \mathfrak{X} \top.\]

Since \(\Box (a \rightarrow \neg \beta(\bar{a})) \rightarrow (\Box a \rightarrow \Box \neg \beta(\bar{a})) = \mathfrak{X} \top\), we have

\[\Box a \rightarrow \Box \neg \beta(\bar{a}) \geq \mathfrak{X} \Box (a \rightarrow \neg \beta(\bar{a})) = \mathfrak{X} \Box \top = \mathfrak{X} \top,\]

hence \(\Box a \rightarrow \Box \neg \beta(\bar{a}) = \mathfrak{X} \top\). Thus \(\Box a \rightarrow \Box \neg \beta(\bar{a})\), \(\Box a \in u\), therefore \(\neg \diamond \beta(\bar{a}) = \mathfrak{X} \Box \neg \beta(\bar{a}) \in u\), this contradicts to \(\Box \beta(\bar{a}) \in u\).

Thus \(X \cup \{\beta(a)\}\) can be extended to an ultrafilter, say \(v\). By Definition 4, \((\mathfrak{X}^*, v) \models \beta(\bar{a})\). Also \(uRv\) holds, because \(X \subseteq v\).

For the other direction, suppose that there exists \(v\) such that \(uRv\) and \(v \models \beta(\bar{a})\), but \(\Box \neg \beta(\bar{a}) = \mathfrak{X} \Box \neg \beta(\bar{a}) \in u\), again by Definition 4, \(\beta(\bar{a}) \in v\), and it follows from \(uRv\) and \(\Box \neg \beta(\bar{a}) \in u\), that \(\neg \beta(\bar{a}) \in v\), a contradiction. ■

The underlying frame of \(\mathfrak{X}^*\) can be quite different from the frame of the model \(\mathfrak{X}\). Nevertheless it retains some important properties of \(\mathfrak{X}\).
Lemma 4
If $\mathfrak{A}$ is an $LTL$-algebra, then for all $u, v, w \in \mathfrak{A}^*$:

1. $uRv & uRw \implies vRw \lor wRv$,
2. $uRv \implies uRw$,
3. $uRv & u \neq v \implies N(u)Rv$.

Proof. (1) Frames that satisfy the condition

$$\forall u, v, w (uRv & uRw \implies vRw \lor wRv)$$

are called weakly connected. It is well known that this condition is preserved under transition from Boolean algebras with operators to models of ultrafilters. (See for details, for instance, [40, Theorem 2.3.51], where the proof uses $LTL$-theorems (T2).)

(2) Suppose $uRv$ and $a \in v$. Then $Na \in u$. Since $N\chi \rightarrow \omega x \in LTL$, then $Na \rightarrow \omega a \in u$ (by Lemma 1). By the properties of ultrafilters, it follows from $Na \rightarrow \omega a \in u$ and $Na \in u$ that $\omega a \in u$. Thus for each $a \in A$, from $a \in v$ it follows that $\omega a \in u$, therefore $uRv$, by definition of $\mathfrak{A}^*$.

(3) Let $u, v \in \mathfrak{A}^*$ be such that $uRv$ and $u \neq v$. Suppose $a \in v$. Since $u \neq v$, there exists $c \in \mathfrak{A}$, such that $c \in v$ and $c \notin u$. Therefore, $a \land c \in v$ and $\omega (a \land c) \in u$. Since $\omega x \rightarrow (\omega \chi \lor \omega x) \in LTL$, $\omega (a \land c) \rightarrow ((a \land c) \lor \omega (a \land c)) \in u$, hence $(a \land c) \lor \omega (a \land c) \in u$. Since $c \notin u$, $a \land c \notin u$, therefore $\omega (a \land c) \in u$ and further $\omega (a \land c) \in N(u)$. Since $\omega (a \land c) \rightarrow \omega a \in N(u)$, $\omega a \in N(u)$. Thus, for every $a \in v$, $\omega a \in N(u)$, hence $N(u)Rv$. \hfill $\blacksquare$

4 Reduced forms of inference rules

A (conjunctive) $n$-clause (or clause if $n$ is clear from context) is a formula over $L$-language with $n$ variables, which has the form

$$\bigwedge_{i=1}^{n} x_{i}^{TB(i)} \land \bigwedge_{i=1}^{n} (N_{x_{i}})^{tN(i)} \land \bigwedge_{i,j=1}^{n} (x_{i}U_{x_{j}})^{tU(i,j)}.$$  (1)

where $tB, tN : \{1, \ldots, n\} \rightarrow \{0, 1\}$ and $tU : \{1, \ldots, n\} \times \{1, \ldots, n\} \rightarrow \{0, 1\}$ are some functions and for each formula $\alpha, \alpha_0 := \neg \alpha, \alpha_1 := \alpha$. It is easy to see that there can be only as many as $2^{2^{n(n+2)}}$ distinct clauses over a set of $n$ variables. We denote the set of all clauses over variables $x_1, \ldots, x_n$ by $\Phi(x_1, \ldots, x_n)$.

For every $\phi \in \Phi(X)$, let

1. $\theta_{S}(\phi) := \{x_i \in X \mid t_S(i) = 1\}$, for $S \in \{B, N\}$;
2. $\theta_{U}(\phi) := \{(x_i, x_j) \in X \times X \mid t_U(i, j) = 1\}$.

We say that a clause $\phi$ is satisfiable in a model $\mathcal{M}$, if there is a world $w \in \mathcal{M}$, such that $\mathcal{M}, w \models \phi$. Given a Kripke model $\mathcal{M}$ with the valuation $v$ of finite domain, every world $w \in \mathcal{M}$ determines a unique clause $\phi_{w}(\mathcal{M}) \in \Phi(\text{dom}(w))$, defined as

$$\phi_{w}(\mathcal{M}) := \bigwedge_{w \vdash x_{i}^{TB}} \bigwedge_{w \vdash (N_{x_{i}})^{tN}} \bigwedge_{w \vdash (x_{i}U_{x_{j}})^{tU}}.$$  

where $e_{TB}, e_{N}, e_{U} \in \{0, 1\}$ are some constants. We will not be mentioning $\mathcal{M}$ in $\phi_{w}(\mathcal{M})$, whenever the model $\mathcal{M}$ is clear from context. A rule $r$, with $\text{Var}(r) = \{x_1, \ldots, x_n\}$, is said to be in the reduced
normal form if

\[ r = \bigvee_{1 \leq j \leq s} \phi_j / x_1, \]

where each disjunct \( \phi_j \) is an \( n \)-clause. For a formula \( \alpha \), \( \text{Sub}(\alpha) \) denotes the set of subformulas of \( \alpha \).

For a rule \( r = \alpha/\beta \): \( \text{Sub}(r) := \text{Sub}(\alpha) \cup \text{Sub}(\beta) \).

**Definition 5**

A Kripke model \( \mathcal{M} = (\mathcal{F}, \mu) \) is a definable variant of a model \( \mathcal{N} = (\mathcal{F}, \nu) \) if for every \( x_i \) from the domain of \( \mu \) there is a formula \( \alpha_i \), with \( \text{Var}(\alpha_i) \subseteq \text{dom}(\nu) \), such that

\[ w \models_{\mu} x_i \iff w \models_{\nu} \alpha_i \]  

(in other words: \( \mu(x_i) = \nu(\alpha_i) \))

We say that two rules \( r_1, r_2 \) are definably equivalent if for every Kripke model \( \mathcal{M} \), based on a frame, if \( \mathcal{M} \not\models r_1 \), then there is a definable variant \( \mathcal{N} \) of \( \mathcal{M} \), with \( \mathcal{N} \not\models r_2 \), where \( i, j \in \{1, 2\} \).

Note, that for two definably equivalent rules \( r_1, r_2 \): \( r_1 \) is admissible in \( \mathcal{L} \mathcal{T} \mathcal{C} \) iff \( r_2 \) is admissible in \( \mathcal{L} \mathcal{T} \mathcal{C} \).

Using the technique similar to one described in [40, Section 3.1], we can transform every inference rule over the language \( \mathcal{L} \) to a definably equivalent rule in the reduced normal form.

**Lemma 5**

Every rule \( r = \alpha/\beta \) can be transformed in exponential time to a definably equivalent rule \( r_{nf} \) in the reduced normal form.

**Proof.** We shall specify for the language \( \mathcal{L} \) the general algorithm described in [Lemma 3.1.3 and Theorem 3.1.11 40].

Let \( r = \alpha/\beta \) be an inference rule. We will need a set of new variables \( Z = \{ \gamma_{\gamma} \mid \gamma \in \text{Sub}(r) \} \).

Let us consider the rule in the intermediate form:

\[ r_{if} = z_{\alpha} \land \bigwedge_{\gamma \in \text{Sub}(r) \setminus \text{Var}(r)} (z_{\gamma} \leftrightarrow \gamma^2) / z_{\beta}, \]

where

\[ \gamma^2 = \begin{cases} z_\delta \ast z_\epsilon & \text{when } \gamma = \delta \ast \epsilon \text{ for } \ast \in \{\land, \lor, \rightarrow, \lnot\}, \\ z_\delta & \text{when } \gamma = \lnot \delta \text{ for } \ast \in \{\land, \lor\}. \end{cases} \]

The rules \( r \) and \( r_{if} \) are definably equivalent. Indeed, suppose \( \mathcal{M} \) is a model with a valuation \( v: \text{Var}(r) \rightarrow 2^W \) over a frame \( \mathcal{F} = (W, R, N) \), such that \( \mathcal{M} \not\models r \). Then \( \mathcal{F} \not\models v(\alpha(x)) \) and there exists an element \( w \in W \), such that \( (\mathcal{F}, w) \not\models \beta(\check{x}) \). Let \( \mu: Z \rightarrow 2^W \) be the valuation defined as follows: \( \mu(\gamma_{\gamma}) = v(\gamma) \). It is straightforward to show that \( \mathcal{F} \models_{\mu} z_{\alpha} \land \bigwedge_{\gamma \in \text{Sub}(r) \setminus \text{Var}(r)} (z_{\gamma} \leftrightarrow \gamma^2) \). In addition, \( (\mathcal{F}, w) \not\models_{\mu} z_{\beta} \).

For the other direction, suppose \( \mathcal{F} \models_{\mu} z_{\alpha} \land \bigwedge_{\gamma \in \text{Sub}(r) \setminus \text{Var}(r)} (z_{\gamma} \leftrightarrow \gamma^2) \) and \( (\mathcal{F}, w) \not\models_{\mu} z_{\beta} \), for some valuation \( \mu: Z \rightarrow 2^W \) and some \( w \in W \). Define \( v: \text{Var}(r) \rightarrow 2^W \) by \( v(x_i) = \mu(z_{\alpha_i}) \). It follows directly that for all \( \gamma \in \text{Sub}(r) \), \( v(\gamma) = \mu(\gamma_{\gamma}) \). Thus \( \mathcal{F} \models v(\alpha(\check{x})) = (\mathcal{F}, w) \not\models \beta(\check{x}) \), hence \( \mathcal{F} \not\models v, r \).

Finally, we transform the premise of the obtained rule \( r_{if} \) into a perfect disjunctive normal form over primitives of the form \( x_i, N x_i \) and \( U x_i \). This requires no more than exponential time on the number of variables, i.e. on the number of subformulas of the original rule (the same as for reduction of any Boolean formula to the perfect disjunctive normal form).
Lemma 5 gives us a procedure, that deterministically assigns a particular normal form to each rule. We will denote this form by \( r_{af} \). As seen from the proof of Lemma 5, the variables in the reduced form represent the subformulas of \( \alpha \) and \( \beta \). In particular, the variable \( x_1 \) (in the future \( x_1 \)) stands for the conclusion \( \beta \) itself.

5 Coherent strings of clauses

**Definition 6**

A finite sequence of clauses \( \phi_1, \phi_2, \ldots, \phi_n \) is called a coherent string if for every \( i < n \):

1. \( \theta_{N}(\phi_i) = \theta_{B}(\phi_{i+1}) \);
2. \( (x,y) \in \theta_{L}(\phi_1) \iff y \in \theta_{B}(\phi_i) \lor (x \in \theta_{B}(\phi_i) \& (x,y) \in \theta_{L}(\phi_{i+1})) \);
3. \( (x,y) \in \theta_{L}(\phi_n) \implies x \in \theta_{B}(\phi_n) \lor y \in \theta_{B}(\phi_n) \).

Let \( \phi_1, \ldots, \phi_n \) be a sequence of clauses (e.g. a coherent string). A variable \( y \) is delayed at \( \phi_1 \) until after \( \phi_j \), where \( 1 \leq j \leq n \), if

\[
\exists x ((x,y) \in \theta_{L}(\phi_1)) \& \forall s (1 \leq s \leq j \implies y \notin \theta_{B}(\phi_s)).
\]

The set of such variables is denoted by Del(\( \phi_1, \phi_i \)). We will also write Del(\( \phi \)) instead of Del(\( \phi, \phi \)) and say that \( x \in \text{Del}(\phi) \) is delayed (at \( \phi \)). It is easy to check that, for every coherent string, Del(\( \phi_1, \phi_j \)) \( \subseteq \) Del(\( \phi_j \)).

A coherent loop is a coherent string \( \phi_1, \phi_2, \ldots, \phi_n, \phi_1 \) such that

\[
\bigcup_{i=1}^{n} \text{Del}(\phi_i) \subseteq \bigcup_{i=1}^{n} \theta_{B}(\phi_i).
\]

Coherence is a syntactic notion, which describes the connection between structures of clauses in a sequence. An obvious source of coherent strings are Kripke models associated with \( \text{LTL} \)-algebras.

**Lemma 6**

For all \( u, v \in \mathfrak{X}^* \)

\[
uNv \implies (\phi_u, \phi_v) \text{ is a coherent string.}
\]

**Proof.** The three conditions of Definition 6 are directly derivable from the following relations between ultrafilters \( u \) and \( v \):

1. \( x \in v \iff N(x) \iff Nx \in u \).
2. \( (\Leftrightarrow) \) Suppose \( xUy \in u \) and \( y \notin u \). By (T5), \( x, N(xUy) \in u \), therefore also \( xUy \in v \), as needed.
3. \( (\Rightarrow) \) If \( y \in u \), then \( \Diamond y \in u \), therefore \( xUy = x \land \Diamond y \land (x \land N(xUy)) \in u \), as needed. If \( x \in u \) and \( xUy \in N(y) = v \), then, in particular, \( \Diamond y \in N(u) \), hence \( y \in u \). Thus \( xUy = x \land \Diamond y \land (x \land N(xUy)) \in u \).
4. By Lemma 2, there exists \( w = N(v) \), and, by (2), \( x \in u \) or \( y \in u \).

As a direct consequence of Definition 6 we obtain the following.

**Lemma 7**

If strings \( \phi_1, \ldots, \phi_k \) and \( \phi_k, \ldots, \phi_{k+1} \) are coherent, then the string \( \phi_1, \ldots, \phi_{k-1}, \phi_k, \phi_{k+1}, \ldots, \phi_{k+1} \) is coherent.
A basis for LTL-admissible rules

**Definition 7**
Let \( \{a_0, ..., a_n\} \) be an \( n+1 \)-element set. A \( b \)-frame (balloon-frame) is a finite frame
\[
\langle \{a_0, ..., a_n\}, N, R \rangle,
\]
such that
1. there is \( k \): \( 0 \leq k \leq n \) such that
\[
N = \{ \langle a_i, a_{i+1} \rangle \mid i = 0, ..., n - 1 \} \cup \{ \langle a_n, a_k \rangle \},
\]
2. \( R \) is the reflexive and transitive closure of \( N \).

A frame \( \langle W \cup \{a\}, N \cup \{(a, a)\}, R \cup \{(a, a)\} \rangle \), where \( \langle W, N, R \rangle \) is a \( b \)-frame and \( a \not\in W \), is called a \( b_e \)-frame.

A Kripke model based on a \( b(b_e) \)-frame (see Figure 1) we will call a \( b(b_e) \)-model, whenever \( \phi_{a_0} = \phi_{a_1} \).

In what follows, we will be dealing with special \( b \)-models, the universes of which are multisets of clauses. The description and the most important property of these models is given in the following lemma.

**Lemma 8**
Let \( \{\phi_0, \phi_0, ..., \phi_k, ..., \phi_{k+l}\} \) be a multiset of clauses and
\[
\mathcal{M} = \langle \{\phi_0, \phi_0, ..., \phi_k, ..., \phi_{k+l}\}, N, R, v \rangle
\]
be a \( b \)-model, where \( \phi_{k+l} \models N \phi_k \) (see Figure 2 for illustration), such that
1. \( v(x) = \{ \phi_i \mid x \in \theta_B(\phi_i) \} \); truth-values for operations \( N \) and \( U \) are defined in the model \( \mathcal{M} \) as in standard LTL-models based on \( N \);
2. \( \phi_0, \phi_0, ..., \phi_k \) is a coherent string; and
3. \( \phi_k, ..., \phi_{k+l}, \phi_k \) is a coherent loop.

Then for all \( \phi_i \): \( (\mathcal{M}, \phi_i) \models \phi_i \).
Lemma 9

where \( \bar{x} \) are

Let \( \langle x, y \rangle \in \Theta_U(\phi_l) \) and we are done. Suppose then that \( j > i \). We will prove by induction, that \( \langle x, y \rangle \in \Theta_U(\phi_s) \), for all \( i \leq s \leq j \).

Let \( s = j \), then \( y \in \Theta_B(\phi_s) \), and Definition 6(2) can be applied to \( \phi_s \), since \( \phi_s \) is the first element of some coherent pair. Therefore \( \langle x, y \rangle \in \Theta_U(\phi_s) \).

Let \( s < j \). We have that \( x \in \Theta_B(\phi_s) \) and, by inductive hypothesis, \( \langle x, y \rangle \in \Theta_U(\phi_{s+1}) \), therefore, by Definition 6(2), \( \langle x, y \rangle \in \Theta_U(\phi_s) \), as needed.

\( (\Leftarrow) \) Suppose \( \langle x, y \rangle \in \Theta_U(\phi_i) \). If \( y \in \Theta_B(\phi_j) \), then \( \phi_j \vdash x \) and therefore \( \phi_i \vdash x \), as needed. Otherwise \( y \) is delayed at \( \phi_i \). Let \( \phi_i, \ldots, \phi_j \) be the shortest sequence of clauses, such that \( y \in \Theta_B(\phi_i) \). Such sequence exists because

\[
\begin{align*}
\text{if } i < k & : \text{Del}(\phi_i, \phi_k) \subseteq \text{Del}(\phi_k) \subseteq \bigcup_{s=0}^{i} \Theta_B(\phi_{k+s}) \\
\text{if } i \geq k & : \text{Del}(\phi_i) \subseteq \bigcup_{s=0}^{i} \Theta_B(\phi_{k+s})
\end{align*}
\]

by properties of coherent strings and loops.

We may show by induction that for all \( s = i, \ldots, j - 1 \): \( x \in \Theta_B(\phi_s) \) (hence \( \phi_i \vdash x \)) and \( \langle x, y \rangle \in \Theta_U(\phi_{s+1}) \). The inductive step for \( U \) is: since \( \langle x, y \rangle \in \Theta_U(\phi_i) \) by inductive hypothesis \( \langle x, y \rangle \in \Theta_U(\phi_i) \) by assumption) and \( y \not\in \Theta_B(\phi_s) \), then, by Definition 6(2), \( x \in \Theta_B(\phi_j) \) and \( \langle x, y \rangle \in \Theta_U(\phi_{s+1}) \).

To be able to construct \( b \)-models of clauses out of Kripke models \( \mathfrak{A}^* \), we also need several combinatorial results.

Definition 8

Suppose we have a finite non-empty set \( A \) and suppose that \( a, b \in A, a \neq b \). A reflexive binary relation \( P \subseteq A \times A \) is called \( (a, b) \)-oriented, whenever, for every subset \( X \subseteq A \), if \( a \in X \) and \( b \not\in X \), then there are \( x, y \in A \), such that \( x \in X \), \( y \not\in X \), and \( xPy \). In symbols:

\[
\forall X \subseteq A : (a, b) \in X \times \bar{X} \implies \exists (x, y) \in P \cap (X \times \bar{X}),
\]

where \( \bar{X} \) is \( A \setminus X \)—the complement of \( X \) in \( A \).

Lemma 9

If \( P \subseteq A^2 \) is \( (a, b) \)-oriented, then there exists a \( P \)-path from \( a \) to \( b \).

Proof. By contradiction. Suppose not and let \( X \) be the set of all elements of \( A \) accessible via \( P \)-paths from \( a \). Since \( P \) is reflexive, \( a \in X \), therefore by assumption \( (a, b) \in X \times \bar{X} \). Thus, there exists \( \langle x, y \rangle \in P \cap (X \times \bar{X}) \), in contradiction with the definition of \( X \).

In \( \mathfrak{A}^* \), we will write \( R(u) \) for the set \( \{ v \mid uRv \} \).

Lemma 10

Let \( \xi, \eta \in \mathfrak{A}^* \). If \( \xi R \eta \), then there exists a coherent string

\[
\phi_\xi, \phi_{u_1}, \ldots, \phi_\eta
\]

of clauses, such that all intermediate \( u_i \)'s, if they exist, are also in \( R(\xi) \).
Let us consider an auxiliary sequence of inference rules

\[ P := \{ \langle \phi_u, \phi_v \rangle \mid u, v \in \mathcal{X} \land u \not\models \phi_v \} \cup \{ \langle \phi_u, \phi_v \rangle \mid u \in \mathcal{X} \}. \]

**Claim 1**

\( P \subseteq A \times A \) is \( \langle \phi_u, \phi_v \rangle \)-oriented.

**Proof.** Let \( \mathcal{X} \subseteq A \) be such that \( \phi_u \in \mathcal{X}, \phi_v \in \bar{\mathcal{X}} \). Let us denote \( \alpha := \bigtriangledown \phi_v \to \bigvee_{\phi \in \mathcal{X}} \phi \). Let us consider the truth-value of the theorem

\[ \gamma := (\alpha \land \square (\alpha \to \mathcal{N} \alpha)) \to \square \alpha \in \mathcal{LTL} \]

(the substitution variant of (T4)) at the ultrafilter \( \xi \in \mathcal{X}^* \).

Since \( \xi \models \gamma, \xi \models \alpha \) and \( \xi \not\models \square \alpha \) (the latter is because \( \xi \mathcal{R} \eta \) and \( \eta \not\models \alpha \)), then \( \xi \not\models \square (\alpha \to \mathcal{N} \alpha) \). Therefore, there exists \( u \in \mathcal{X}^* \), such that \( \xi \mathcal{R} u \) and \( u \not\models \alpha \to \mathcal{N} \alpha \), hence \( u \models \alpha \) and \( u \not\models \mathcal{N} \alpha \). This means that \( u \models \bigtriangledown \phi_v \to \bigvee_{\phi \in \mathcal{X}} \phi, N(u) \models \bigtriangledown \phi_v \), and \( N(u) \not\models \bigvee_{\phi \in \mathcal{X}} \phi \). Thus, both \( u, N(u) \in \mathcal{X} \), \( \langle \phi_u, \phi_{N(u)} \rangle \in P \), but \( \phi_u \in \mathcal{X} \) and \( \phi_{N(u)} \in \bar{\mathcal{X}} \). Therefore, \( \langle \phi_u, \phi_{N(u)} \rangle \in P \cap (\mathcal{X} \times \bar{\mathcal{X}}) \).

It follows from Lemma 9, that there exists a sequence \( \phi_1, \phi_2, \ldots, \phi_n \) of clauses from \( A \), with \( \phi_1 = \phi_v \), \( \phi_n = \phi_v \), such that for each pair \( \phi_i, \phi_{i+1} \) there are ultrafilters \( u, v \in \mathcal{X}^* \), with the property \( v \models N(u) \). By Lemma 7, each pair \( \phi_i, \phi_{i+1} \) forms a coherent string. Using Lemma 6, we can “stitch” together the coherent strings \( \phi_1, \phi_2; \phi_2, \phi_3; \ldots; \phi_{n-1}, \phi_n \) to obtain the coherent string \( \phi_k = \phi_1, \ldots, \phi_n = \phi_v \), where \( \phi_1 = \phi_v \) and \( \phi_n = \phi_v \).

## 6 Main results

Let us consider an auxiliary sequence of inference rules \( s_2, s_3, \ldots \)

\[ s_n(x_1, \ldots, x_n) := \left[ \bigvee_{i=1}^{n} x_i \land \bigwedge_{i \neq j} \neg(x_j \land x_i) \land \left( \bigwedge_{i=1}^{n} x_i \to \bigvee_{j=i}^{n} x_j \right) \right] \]

All rules \( s_n, n = 2, 3, \ldots \), are admissible for \( \mathcal{LTL} \). To show that, it suffices to prove that for every set of \( \mathcal{LTL} \)-formulas \( \alpha_1, \ldots, \alpha_n \), the formula \( \gamma := (\bigwedge_{i=1}^{n} \alpha_i) \land \bigwedge_{i \neq j} \neg(\alpha_i \land \alpha_j) \land \bigwedge_{i=1}^{n} (\alpha_i \to \bigvee_{j \neq i} \alpha_j) \) is not a theorem of \( \mathcal{LTL} \) or, equivalently, that \( \mathcal{M} \not\models \gamma \). Indeed, suppose that \( \mathcal{M} \models (\bigwedge_{i=1}^{n} \alpha_i) \land \bigwedge_{i \neq j} \neg(\alpha_i \land \alpha_j) \) and let \( \mathcal{M} = (\mathcal{N}, v) \) be a direct summand of \( \mathcal{M} \) with a constant valuation \( v \) (i.e. \( v(x) \in \{ \emptyset, \mathcal{N} \} \) for all variables \( x \) from the domain of \( v \)). Then, for some \( \alpha_i, \mathcal{M} \models \alpha_i \), therefore \( \mathcal{M} \not\models \alpha_i \to \bigvee_{j \neq i} \alpha_j \), hence \( \mathcal{M} \not\models \gamma \).

**Lemma 11**

Let \( \mathcal{A} = \mathcal{A}(\bar{a}) \) be an \( n \)-generated \( \mathcal{LTL} \)-algebra, such that \( \mathcal{A} \models s_i \), for all \( i = 2, 3, \ldots, 2^{n(n+2)} \). Then there is \( u \in \mathcal{A}^* \), such that

1. \( \theta(\phi_u) = \theta(\phi_u) \);
2. \( \mathcal{N}(\phi_u) = \emptyset \).

**Proof.** Let \( \phi_1, \ldots, \phi_k \) be an enumeration of all distinct elements from the set \( \{ \phi_u \mid u \in \mathcal{A}^* \} \). If \( k = 1 \), then \( \phi_u = \phi_1 \) and the proof is straightforward. Otherwise \( \mathcal{A}^* \models \bigvee_{i=1}^{n} \phi_i, \mathcal{A}^* \models \bigwedge_{i \neq j} \neg(\phi_i \land \phi_j) \). Since \( k \) cannot be bigger than the number of \( n \)-clauses, i.e. \( k \leq 2^{n(n+2)} \), then \( \mathcal{A}^* \not\models s_k \), therefore \( \mathcal{A}^* \not\models \bigwedge_{i=1}^{n} (\phi_i \to \bigvee_{j \neq i} \phi_j) \). Thus, there is \( \phi_u \) and \( u \in \mathcal{A}^* \), such that \( u \models \phi_i \), and \( u \not\models \bigvee_{j \neq i} \phi_j \), hence \( u \not\models \bigtriangledown \phi_i \).
(1) By Lemma 2, there is $N(u) \in \mathbb{A}^*$, for which, by Lemma 4(2), $uRN(u)$ holds. Thus $\phi_{N(u)} = \phi_i$, thereby $\theta_B(\phi_u) = \theta_N(\phi_u).

(2) Suppose $y$ is a variable such that $\langle x, y \rangle \in \theta_U(\phi_u)$, for some $x$. Then there exists $v$ such that $uRv$ and $v \models y$, therefore $y \in \theta_B(\phi_i)$. But $\phi_v = \phi_i$, hence $y$ is not delayed. ■

From the obtained in Lemma 11 clause $\phi_i$ we can construct an one-element Kripke model $e:=(\langle \phi_i \rangle, \{\phi_i, \phi_j\}, \{\phi_i, \phi_j\}, v)$, where $\nu(x_i) = \{\phi_i\}$, if $x_i \in \theta_B(\phi_i)$, and $\nu(x_i) = \emptyset$, if $x_i \notin \theta_B(\phi_i)$. Clearly, $\nu \models \phi_i$ and we will use $e$ in the following, as ‘$e$-part’ of a $b_e$-model.

Let, for $n=1,2,\ldots$,

$$r_n(x_0,x_1,\ldots,x_n) := \bigwedge_{i=1}^n (x_i \leftrightarrow \Box x_i / x_0).$$

All rules $r_n$ are admissible in $\mathcal{LTL}$. Indeed, let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be some $\mathcal{LTL}$-formulas. Suppose $\mathcal{L} \models \alpha_0$. It means that there exists $N_0$ and $i \in N_0$, such that $(N_0, i) \not\models \alpha_0$. Without loss of generality, we can assume that $i = 0$. For proving admissibility, it suffices to show that $\bigwedge_{i=1}^n (a_i \leftrightarrow \Box a_i) \rightarrow \square a_0 \not\models \mathcal{L}$. Or, equivalently, that there exists $N_{10}$, such that $(N_{10}, 0) \not\models \bigwedge_{i=1}^n (a_i \leftrightarrow \Box a_i) \land \neg a_0$. Let $m := \max \{\deg(\alpha_i) | a_1, \ldots, a_n\}$, where $\deg(\alpha)$ is the modal degree of $\alpha$. We can take as $N_{10}$ a model, such that (1) $N_{10}[m, \infty]$ is isomorphic to $N_0$; (2) first $m$ elements $0, \ldots, m-1$ of $N_{10}$ have the same valuation of variables.

**Lemma 12**

Let $r = \bigvee_{1 \leq j \leq k} \phi_j(x_1,\ldots,x_n)/x_1$ be a rule in the reduced form and suppose that $\mathcal{A} = \mathcal{A}(a_1,\ldots,a_n)$ is an $n$-generated $\mathcal{LTL}$-algebra, such that

1. $\mathcal{A} \not\models \bigvee_{1 \leq j \leq k} \phi_j(\bar{a})/a_1;
2. \mathcal{A} \models r_k$.

Then there exists a $b$-model $\mathcal{M}$ such that $\mathcal{M} \not\models r$.

**Proof.** By assumption,

$$\mathcal{A} \models \bigvee_{1 \leq j \leq k} \phi_j(\bar{a}), \ \mathcal{A} \not\models a_1.$$

Consider the valuation (on algebra $\mathcal{A}$) for variables of $r_k$:

$$x_0 \mapsto a_1, \ \ x_j \mapsto \phi_j(\bar{a}), \ \text{for } j = 1,\ldots,k.$$

Since $\mathcal{A} \models r_k$ and $\mathcal{A} \not\models a_1$, then $\mathcal{A} \not\models \bigvee_{1 \leq i \leq k} (\phi_i(\bar{a}) \leftrightarrow \Box \phi_i(\bar{a})) \rightarrow \square a_1$, hence, by Lemma 1(1), there must be $\xi \in \mathbb{A}^*$ such that

$$\bigvee_{1 \leq i \leq k} (\phi_i(\bar{x}) \leftrightarrow \Box \phi_i(\bar{x})) \land \neg a_1.$$

Therefore, since $\phi_1 \in \{\phi_1,\ldots,\phi_k\}$, then $\phi_1 = \phi_1(\bar{x})$ and $\neg a_1 \in \xi$. It follows from $\land \neg a_1 \in \xi$ that there exists $\mu \in \mathbb{A}^*$, such that $x_1 \in \mathcal{M}$, therefore, $(\mathcal{A}, \xi) \models \bigvee_{1 \leq i \leq k} (\phi_i(\bar{x}) \leftrightarrow \Box \phi_i(\bar{x})) \land \neg a_1$.

For every $u \in \mathbb{A}^*$, we define

$$D^+(u) := \{\phi_0 | uRv\}.$$

Recall also that we write $R(u)$ for the set $[v | uRv]$. 

---

A basis for LTL-admissible rules

It follows directly, that for every $u \in \mathfrak{A}^*$: $\emptyset \neq D^+(u) \subseteq \{\phi_1, \ldots, \phi_k\}$ and $D^+(v) \subseteq D^+(u)$, whenever $uRv$. Let

$$A := D^+(N(\xi));$$

$$\tilde{N} := \{(\phi_u, \phi_v) | u, v \in R(N(\xi)) \land u \sim v \};$$

$$\tilde{R} := \tilde{N}^*, \text{ (i.e. } \tilde{R} \text{ is a reflexive and transitive closure of } \tilde{N}).$$

By definition, $\langle A, \tilde{R} \rangle$ is a finite non-empty preorder. In addition the following holds.

**Claim 2**

The preorder $\langle A, \tilde{R} \rangle$

(1) is linear, i.e., $\forall \psi \exists \phi \phi \tilde{R}\psi \lor \psi \tilde{R}\phi$;

(2) has a source, i.e., $\exists \phi \forall \psi \phi \tilde{R}\psi$,

(3) has a drain, i.e., $\exists \phi \forall \psi \psi \tilde{R}\phi$.

**Proof.** (1) Let $\phi, \psi \in A$. By definition of $A$, there exist $u, v \in \mathfrak{A}^*$, such that $u, v \in R(N(\xi)), \phi_u = \phi$, and $\phi_v = \psi$. By Lemma 4(1), $\mathfrak{A}^*$ is weakly connected, therefore $uRv$ or $vRu$, hence, by Lemma 10, $\phi \tilde{R}\psi$ or $\psi \tilde{R}\phi$.

(2–3) Since $\langle A, \tilde{R} \rangle$ is a linear, finite, non-empty preorder, it has the least and largest clusters (they might coincide). For a source we choose any element from the least, and for a drain an element from the largest cluster.

Let $\eta \in R(N(\xi))$ be such an ultrafilter that $\phi_\eta$ is a drain of $\langle A, \tilde{R} \rangle$. All clauses from the set $D^+(\eta)$ form a cluster in $\langle A, \tilde{R} \rangle$. So, we can form a $\tilde{R}$-path containing all elements from $D^+(\eta)$, possibly with repetitions, and then expand it to a coherent string $L = \phi_{\eta_1}, \ldots, \phi_{\eta_t}$, using equality $R = N^*$. To prove that $L$ is a coherent loop, suppose $y \in \text{Del}(\phi)$ for some $\phi \in D^+(\eta)$. Then there is a $u \in R(\eta)$, such that $\phi_u = \phi$, therefore $xUy \in u$. By (T5), $\diamond y \in u$ and, by Lemma 3, there exists $v$ such that $uRv$ and $y \in v$. Thus $\phi_v \in D^+(\eta)$, $\phi_v \tilde{R}\phi_u$, and $y \in \theta_B(\phi_u)$.

Finally, by stitching together all obtained coherent parts: $\phi_{\xi}, \phi_{\xi}; \phi_{\xi}, \ldots, \phi_{\eta_t}$; and attaching to the result the final coherent loop $\phi_{\eta_1}, \ldots, \phi_{\eta_t}$ at the point $\phi_{\eta_t}$, we obtain the $b$-model (with the universe as a multiset of clauses), that refutes $r$, by Lemma 8.

**Lemma 13**

Let $r = \bigvee_{1 \leq j \leq s} \phi_j(x_1, \ldots, x_n) / x_1$ be a rule in the reduced form and suppose there is a $b_c$-model $M = \langle \mathcal{F}, \mu \rangle$ with $\text{dom}(\mu) = \{x_1, \ldots, x_n\}$, such that $M \not \models r$. Then $r$ is not admissible in $\mathcal{LTL}$.

**Proof.** Re-enumerating the disjuncts of $r$ if necessary, we can assume that $M$ has the form as in Figure 2.

**Figure 2. Model $M$.**
Claim 3
Without loss of generality, we can assume that
\[ \langle \phi_{k+i}, \ldots, \phi_{k+i+l} \rangle \neq \langle \phi_{k+j}, \ldots, \phi_{k+j+l} \rangle, \]
whenever \( i \neq j \), where \([i] := i \mod (l+1)\).

Proof. Suppose \( \langle \phi_{k+i}, \ldots, \phi_{k+i+l} \rangle = \langle \phi_{k+j}, \ldots, \phi_{k+j+l} \rangle \) and let \( i < j \). If \( j - i \leq l/2 \), we can simply cut \( \phi_{k+i}, \ldots, \phi_{k+j} \) out of the loop, which will not change the truth-values of clauses at the remaining elements. If \( j - i > l/2 \), then, by assumption \( \langle \phi_{k+i}, \ldots, \phi_{k+i+l} \rangle = \langle \phi_{k+j}, \ldots, \phi_{k+j+l} \rangle \), we have that the paths from \( \phi_{k+j} \) to \( \phi_{k+i} \) and from \( \phi_{k+i} \) to \( \phi_{k+i+1-j+2l} \) coincide. So we can delete one of these paths (which has no entry point from \( \phi_{k-1} \)). This transformation again will not affect truth-values of clauses at the remaining elements. By iterating this procedure, we will come to the loop with no repetitions, while still satisfying conditions of the lemma on \( \mathcal{M} \)-model. ■

Let us introduce auxiliary formulas. For \( i=0, \ldots, l \)
\[ \gamma_{k+i} := \phi_{k+i} \land N \phi_{k+i+1} \land \cdots \land N^l \phi_{k+i+l} \land N^{l+1} \phi_{k+i}, \]
\[ \lambda := \Box \left( \bigvee_{i=0}^l \gamma_{k+i} \right) \land \Box \left( \bigwedge_{i=0}^l \gamma_{k+i} \right). \]

In view of Claim 3, for \( i \neq j \),
\[ \mathcal{M} \models \mu \gamma_i \land \gamma_j \rightarrow \bot \]  \((*)\)
where \( \mathcal{M} \) is the \( n \)-characterizing model as in Definition 1 and \( \mu \) is the valuation of variables \( x_1, \ldots, x_n \) on \( \mathcal{M} \) (as before we will usually omit \( \mu \) in \( \mathcal{M} \models \)).

Let us denote
\[ \Delta_{k+l} := \gamma_{k+l} \land \lambda, \]
\[ \Delta_{k+l-1} := (\gamma_{k+l-1} \land \lambda) \lor (N \Delta_{k+l} \land \neg \lambda), \]
\[ \Delta_{k+l-2} := (\gamma_{k+l-2} \land \lambda) \lor (N \Delta_{k+l-1} \land \neg \lambda), \]
\[ \vdots \]
\[ \Delta_k := (\gamma_k \land \lambda) \lor (N \Delta_{k+1} \land \neg \lambda), \]
\[ \Delta_{k-1} := N \Delta_k \land \neg \lambda, \]
\[ \Delta_{k-2} := N \Delta_{k-1}, \]
\[ \vdots \]
\[ \Delta_2 := N \Delta_3, \]
\[ \Delta_1 := N \Delta_2, \]
\[ \Delta_0 := \Diamond \Delta_1 \land \neg \Delta_1, \]
\[ \Delta_{-1} := \neg \Diamond \lambda. \]

For notational convenience, let also \( \gamma_{k+l+1} := \gamma_k, \Delta_{k+l+1} := \Delta_k \).
CLAIM 4
The following holds:

1. \( \mathfrak{M} \models \lambda \rightarrow N\lambda; \)
2. \( \mathfrak{M} \models \Delta_i \rightarrow \Diamond \lambda, \) for all \( i = 0, \ldots, k+l; \)
3. \( \mathfrak{M} \models (\gamma_{k+i} \land \lambda) \rightarrow N(\gamma_{k+i+1} \land \lambda), \) for all \( i = 0, \ldots, l; \)
4. \( \mathfrak{M} \models \Delta_i \rightarrow N\Delta_{i+1}, \) for all \( i = 1, \ldots, k+l; \) and
5. \( \mathfrak{M} \models \Delta_0 \rightarrow N(\Delta_0 \lor \Delta_1). \)

PROOF. (1) holds since \( \lambda \) is equivalent to a \( \Box \)-formula.

(2) By induction on \( i = k+l, \ldots, 0. \) If \( w \models \Delta_{k+j}, \) then \( w \models \lambda, \) hence \( w \models \Diamond \lambda \) and we are done. Suppose now \( w \models \Delta_i, \) for \( 0 \leq i < k+l. \) Then either \( w \models \gamma_1 \land \lambda, \) hence \( w \models \Diamond \lambda, \) or \( w \not\models N\Delta_{i+1}, \) therefore \( N(w) \not\models \Delta_{i+1}. \) Then \( w \models \Diamond \lambda \) follows from the inductive hypothesis.

(3) If \( w \not\models \gamma_{k+i}, \) then \( N(w) \not\models \phi_{k+i+1} \land 
\ldots \land \phi_{k+i+l}, \) since \( N(w) \not\models \gamma_{k+i+1}, \) in view of Claim 3 and (1).

(4) If \( w \not\models \gamma_{k+i} \land \lambda, \) then, by (3), \( N(w) \not\models \gamma_{k+i+1} \land \lambda, \) hence \( N(w) \models \Delta_{k+i+1}. \) If \( w \not\models N\Delta_{k+i+1} \land \neg \lambda, \) then trivially \( N(w) \models \Delta_{k+i+1}. \)

(5) Suppose \( w \not\models \Diamond \Delta_1 \land \neg \Delta_1 \) and \( N(w) \models \neg \Delta_1. \) Then, since \( N(w) \models \Diamond \Delta_1, \) \( N(w) \models \Delta_0. \)

Let us denote \( \mu(\alpha) := \{ w \in \mathfrak{M} | w \models \mu(\alpha) \}. \)

CLAIM 5
Sets \( \mu(\Delta_1), \ldots, \mu(\Delta_{k+i+1}) \) form a non-trivial partition of the universe of \( \mathfrak{M}, \) i.e.

1. \( \mu(\Delta_1) \neq \emptyset, \) for \( i = -1, \ldots, k+l; \)
2. \( \mathfrak{M} \models \neg \Delta_1 \land \Delta_2, \) whenever \( i \neq j; \)
3. \( \mathfrak{M} \models \bigvee_{i=-1}^{k+l} \Delta_i. \)

PROOF. (1) Given a valuation \( v : [\bar{x}] \rightarrow \mathbb{N}, \) let us denote for every \( j \in \mathbb{N} : v^j(\bar{x}) := \{ x_i | j \in v(x_i) \}. \) Consider models \( \mathcal{N}_{v_1} \) and \( \mathcal{N}_{v_2}, \) where \( (1) v^0(0) = \theta_B(\phi_0), \) \( v^1(1) = \theta_B(\phi_1), \) \( v^2(2) = \theta_B(\phi_2), \ldots, \)
\( v^k(k) = \theta_B(\phi_k), \) \( v^k(k+1) = \theta_B(\phi_{k+1}), \ldots, \)
\( v^k(k+1+i) = \theta_B(\phi_{k+i}), \) for all \( i \in \mathbb{N}. \) By induction, \( (\mathcal{N}_{v_1}, 0) \models \phi_0; (\mathcal{N}_{v_1}, i) \models \phi_{i+1}, \) for \( i = 1, \ldots, k; \) and \( (\mathcal{N}_{v_1}, k+1+i) \models \phi_{k+i}, \) for \( i \in \mathbb{N}. \) Also \( (\mathcal{N}_{v_2}, i) \models \phi_{i+1}, \) for all \( i \in \mathbb{N}. \) Since \( \mathfrak{M} \) contains as open submodels models \( \mathcal{N}_{v_1} \) and \( \mathcal{N}_{v_2}. \) It follows directly from the definitions of \( \Delta_i \)’s, that \( (\mathcal{N}_{v_1}, i) \models \phi_i \) implies \( (\mathcal{N}_{v_1}, i) \models \Delta_i, v \in \{ v_1, v_2 \}. \)

(2) follows from the following cases:

1. Suppose \( w \models \Delta_{-1} \land \Delta_i, \) where \( i \in [0, \ldots, k+l]. \) By Claim 4(2), \( \mathfrak{M} \models \Delta_i \rightarrow \Diamond \lambda, \) therefore \( \mathfrak{M} \models \neg \Delta_1 \land \Box \lambda, \) a contradiction.

2. Suppose \( w \models \Delta_{k+i} \land \Delta_{k+j}, \) where \( i \in [0, \ldots, l-1]. \) It follows from \( w \models \Delta_{k+l} \) that \( w \models \lambda. \) Therefore \( w \models \gamma_{k+l} \land \phi_{k+l}, i \neq l, \) a contradiction to (*).

3. Suppose \( w \models \Delta_{k-1} \land \Delta_{k+l}, \) then \( \mathfrak{M} \models \neg \lambda \land \lambda, \) a contradiction.

4. Suppose \( w \models \Delta_{k+i} \land \Delta_{k+j}, \) where \( i \in [-1, \ldots, l], j \in [0, \ldots, l-1] \) and \( i \neq j. \) In view of Claim 4(4), there is a sequence \( w, N(w), \ldots, N^l(w) \) of the length not exceeding \( l+1, \) such that \( N^i(w) \models \Delta_{k+i} \land \Delta_p, p \neq k+l, \) a contradiction to case (2).

5. Suppose \( w \models \Delta_i \land \Delta_j, \) where \( i, j \in [1, \ldots, k+l] \) and \( i \neq j. \) By (3) and (4), we can assume that at least one of the indices, say \( i \) is less than \( k-1. \) In view of Claim 4(4), there is a sequence \( w, N(w), \ldots, N^i(w) \) of the length not more than \( k-1, \) such that \( N^i(w) \models \Delta_{k-1} \land \Delta_p, p > k-1, \) a contradiction to either case (3) or (4).
Finally, suppose \( w \models \Delta_0 \land \Delta_1 \), where \( i \in \{1, \ldots, k+1\} \). Since \( w \models \Diamond \Delta_1 \) and in view of Claim 4(4), there is a sequence \( w, N(w), \ldots, N^i(w) \) of length more than one, such that \( N^i(w) \models \Delta_1 \land \Delta_p \), \( p > 1 \), a contradiction to case (5).

(3) If \( w \models \neg \Diamond \lambda \), then \( w \models \neg \Delta_i \) and we are done. So assume \( w \models \Diamond \lambda \). If \( w \models \lambda \), then \( w \models \gamma_{k+1} \) for some \( i \in \{0, \ldots, l\} \), therefore \( w \models \gamma_{k+i} \land \lambda \), hence \( w \models \Delta_{k+i} \). Let us assume now that \( w \models \neg \lambda \). Since \( w \models \Diamond \lambda \), there is a sequence \( w, N(w), \ldots, N^i(w) \) such that \( w \models \neg \lambda, N(w) \models \neg \lambda, \ldots, N^{i-1}(w) \models \neg \lambda \) and \( N^i(w) \models \lambda \).

As above, \( N^i(w) \models \Delta_i \) for some \( i \in \{k, \ldots, k+1\} \). We will show by induction on \( s = t - 1, \ldots, 0 \), that \( N^s(w) \models \Delta_{i-s} \) for some \( t_s \in \{0, \ldots, k+1\} \).

Let \( s = t-1, \ldots, t-i \). Inductively, \( N^s(w) \models \neg \Delta_{i-(t-s)-1} \), therefore \( N^s(w) \models \neg \Delta_{i-(t-s)} \).

Let \( s = t-(i+1), \ldots, 0 \). If \( s = t-(i+1) \), then, from the previous step, \( N^s(w) \models \neg \Delta_1 \). Also \( N^s(w) \models \neg \Delta_1 \), since otherwise \( N^s(w) \models \neg \Delta_2 \), which contradicts to (2). So \( N^s(w) \models \Delta_0 \). Further by induction, \( N^s(w) \models \Delta_0 \), and \( N^s(w) \models \neg \Delta_1 \), since otherwise \( N^{s+1}(w) \models \Delta_0 \land \Delta_2 \), which contradicts to (2). So \( N^s(w) \models \Delta_0 \).

Thus, we have \( w \models \Delta_0 \) or, if the sequence cuts short, \( w \models \Delta_{i-t} \).

For \( i = 1, \ldots, n \), let

\[
\alpha_i := \bigvee_{j=1}^{k+1} (T^e_{ij} \land \Delta_j),
\]

where \( e_{ij} = \chi_{\theta_{B}(\phi_j)}(x_i) \) is the characteristic function of the set \( \theta_{B}(\phi_j) \).

Claim 6

Let \( w \models \Delta_i \). Then

1. \( w \models \alpha_i \iff x_i \in \theta_{B}(\phi_i) \).
2. \( w \models \neg \alpha_i \iff x_i \notin \theta_{B}(\phi_i) \).
3. \( w \models \alpha_i \cup \alpha_j \iff \{x_i, x_j\} \in \theta_{B}(\phi_i) \).

Proof. Let \( \Delta_i \in \{\Delta_{i-1}, \ldots, \Delta_{k+1}\} \).

1. Suppose \( w \models \alpha_i \). By Claim 5(2), \( w \models T^{e_{ij}} \land \Delta_i \), hence \( e_{ij} = \chi_{\theta_{B}(\phi_j)}(x_i) = 1 \). For the other direction: suppose \( x_i \in \theta_{B}(\phi_i) \), then \( e_{ij} = 1 \), therefore \( w \models T^{e_{ij}} \land \Delta_i \), hence \( w \models \alpha_i \).

2. There are three cases:

   Case 1: \( t = 1, \ldots, k+1 \). According to Claim 4(4), \( N(w) \models \Delta_{i+1} \). If \( w \models \neg \alpha_i \), then \( N(w) \models \alpha_i \), hence, by (1), \( x_i \in \theta_B(\phi_{i+1}) = \theta_B(\phi_i) \). For the other direction, let \( x_i \in \theta_B(\phi_i) \). Then \( x_i \in \theta_B(\phi_{i+1}) \), and, by (1), \( N(w) \models \alpha_i \), therefore \( w \models \neg \alpha_i \).

   Case 0. Suppose \( w \models \alpha_i \). Then \( N(w) \models \alpha_i \). According to Claim 4(5), \( N(w) \models \Delta_1 \land \Delta_0 \), hence, by (1), \( x_i \in \theta_B(\phi_1) = \theta_B(\phi_0) \) or \( x_i \in \theta_B(\phi_1) = \theta_B(\phi_0) \). For the other direction, let \( x_i \in \theta_B(\phi_0) \). Then \( x_i \in \theta_B(\phi_1) = \theta_B(\phi_0) \). By Claim 4(5), \( N(w) \models \Delta_0 \) or \( N(w) \models \Delta_1 \), so (1) can be applied, therefore \( N(w) \models \alpha_i \) and \( w \models \neg \alpha_i \).

   Case \(-1 \). It follows from \( N(w) \models \neg \Diamond \lambda \) that \( N(w) \models \Delta_{-1} \). Since \( \theta_B(\phi_{-1}) = \theta_B(\phi_{-1}) \) and in view of (1),

\[
N(w) \models \alpha_i \iff \alpha_i \iff x_i \in \theta_B(\phi_{-1}) \iff x_i \in \theta_B(\phi_{-1}).
\]

3. We split the proof into two cases:

   Case 1: \( t = 0, \ldots, k+1 \).

   (⇒) Suppose \( w \models \alpha_i \cup \alpha_j \). Let us choose the smallest \( p \) such that \( N^p(w) \models \alpha_i \). According to Claim 4(4–5), for every \( s \geq 0 \), there exists \( t_s \geq t \) (\( t_0 = t \)), such that \( N^{t_s}(w) \models \Delta_{t_s} \). We will prove by induction, that \( x_i, x_j \in \theta_{B}(\phi_{t_s}) \), for \( s = p, p-1, \ldots, 0 \).
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Let \( s = p \), then, by (1), \( x_j \in \theta_B(\phi_p) \), and Definition 6(2) can be applied to \( \phi_p \). Therefore \( \langle x_j, x_j \rangle \in \theta_U(\phi_p) \).

Let \( s < p \). Since \( N^d(w) \models \alpha_j \), by (1), we have that \( x_j \in \theta_B(\phi_p) \) and, by inductive hypothesis, \( \langle x_j, x_j \rangle \in \theta_U(\phi_{p+1}) \), therefore, by Definition 6(2), \( \langle x_j, x_j \rangle \in \theta_U(\phi_p) \), as needed.

\((\Rightarrow)\) Suppose \( \langle x_j, x_j \rangle \in \theta_U(\phi_p) \). According to Lemma 8, there is a successive sequence of clauses \( \phi_1, \ldots, \phi_p \) from the model \( M \), such that \( x_j \in \theta_B(\phi_p) \) and \( x_j \in \theta_B(\phi_k) \), for all \( \phi_k \) before \( \phi_p \). By Claim 4(4–5), there is a sequence of worlds \( w, N(w), \ldots, N^d(w) \) such that \( w \models \Delta_1, \ldots, N^d(w) \models \Delta_p \), therefore by (1), \( w \models \alpha_1, \ N(w) \models \alpha_1, \ldots, N^d(w) \models \alpha_j \), as needed.

Case \( u = 1 \). Suppose \( \models \Delta_{-1} \). Since Del(\(\phi_{-1}\)) = \(\emptyset\), \( \langle x_j, x_j \rangle \in \theta_U(\phi_{-1}) \) implies \( x_j \in \theta_B(\phi_{-1}) \). Therefore,

\[ \langle x_j, x_j \rangle \in \theta_U(\phi_{-1}) \implies x_j \in \theta_B(\phi_{-1}) \implies w \models \alpha_j \implies w \models \alpha_j U \alpha_j. \]

For the other direction, suppose \( w \models \alpha_j U \alpha_j \). Then there is a sequence \( w, N(w), \ldots, N^d(w) \) such that \( N^d(w) \models \alpha_j \). As above, \( N^d(w) \models \Delta_{-1} \), hence, by (1), \( x_j \in \theta_B(\phi_{-1}) \), therefore \( \langle x_j, x_j \rangle \in \theta_U(\phi_{-1}) \).

It follows directly from Claim 6 that for all \( w \in \mathfrak{N} \):

\[ w \models \Delta_{-1} \implies w \models \phi_j(\alpha_1, \ldots, \alpha_n). \]

Therefore, by Claim 5(3), \( w \models \lor_{f=1}^j \phi_j(\bar{a}) \), while, according to Claim 5(1), \( \mathfrak{N} \not\models \alpha_1 \), hence \( r \) is not admissible in LTL.

It is possible to replace the series \( \{s_n | n \geq 2\} \) with just one rule \( \diamond x \land \diamond \neg x \downarrow \). Indeed, for one direction, the premise of \( \diamond x \land \diamond \neg x \downarrow \) is equivalent modulo LTL to the premise of the substitution variant \( s_2(\diamond x, \land x) \). For the other direction, let \( \mathfrak{A} \) be a LTL-algebra. Clearly, \( \mathfrak{A} \) in the signature \( \langle \lor, \land, \neg, \diamond \rangle \), is an S4-algebra. Let \( \mathfrak{A} \models \diamond x \land \diamond \neg x \downarrow \). Since \( \diamond x \land \diamond \neg x \downarrow \) is equivalent over LTL to the rule \( \diamond x \land \diamond \neg x \downarrow y \), we can employ Lemma 4.3.18 from [40], which states that for any finitely generated S4-algebra \( \mathfrak{A}_1 \), such that \( \mathfrak{A}_1 \models \diamond x \land \diamond \neg x \downarrow y \), there is a single-element R-maximal cluster \( \{u\} \subseteq \mathfrak{A}_1^+ \). Thus, for every \( n = 2, 3, \ldots, \) the premise of \( s_n \) fails on all finitely generated S4-subalgebras of \( \mathfrak{A} \) (as all formulas of these rules are modal ones), therefore vacuously \( \mathfrak{A} \models s_n \).

**Theorem 1**

The set of inference rules

\[ R := \{r_n | n \geq 1\} \cup \{ \diamond x \land \diamond \neg x \downarrow, x, x \rightarrow y \} \]

plus any axiomatization of LTL form a basis for admissible rules of LTL.

**Proof.** As we showed above, all rules \( r_n \) and \( s_k \) are admissible in LTL, for \( x/\Box x \) and \( x, x \rightarrow y/y \) it is evident.

Logic LTL, taken with its postulated inferences rules: \( x/\Box x \) and \( x, x \rightarrow y/y \), falls in the category of (finitely) algebraizable deductive systems [2] (they are called algebraic logics in [40, Definition 1.3.7]). Therefore, we can use the connection between inference rules and quasi-identities (c.f. [40, Theorem 1.4.15]) to establish a basis for admissible rules.

Suppose a rule \( r(\bar{x}) = \alpha(\bar{x})/\beta(\bar{x}) \) does not follow from \( R \). Then there exists a finitely generated LTL-algebra \( \mathfrak{A}(\bar{a}) \), such that \( \mathfrak{A} \models R \), \( \mathfrak{A} \not\models r(\bar{a}) \) (the latter means that \( \mathfrak{A} \models \alpha(\bar{a}) \) and \( \mathfrak{A} \not\models \beta(\bar{a}) \)).

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The authors are thankful to E. Jeřábek for pointing out a way to show this. Actually, he drew our attention to the fact that the rules \( s_n \) are pure modal ones.
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By Lemma 1, $\bigwedge_i \phi_i(x)/x_1$ is definably equivalent to $r$ (where $\vec{x} = (x_1, \ldots, x_n)$ can be different from $\vec{x} = (x_1, \ldots, x_n)$), then for some $\beta_1(x), \ldots, \beta_k(x)$: $\bigwedge_i \phi_i(x)/x_1$ is definably equivalent to $r$. Let $b_1 = \beta_1(\vec{a}), \ldots, b_k = \beta_k(\vec{a})$. On the finitely generated subalgebra $\mathfrak{B} := \mathfrak{A}(\vec{b})$, but $\not\models \rho$, since $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. By Lemmas 11 and 12, there is a $d$-model refuting $r$. Therefore, by Lemma 13, $r$ is not admissible in $\mathcal{LTL}$. Since, by Lemma 5, the rules $r$ and $r$ are definably equivalent, then $r$ is also not admissible in $\mathcal{LTL}$.

7 Conclusions

In this article, we presented an explicit basis for rules admissible in $\mathcal{LTL}$. This, in particular, gives another solution (cf. [44]) to the problem of recognizing rules admissible in $\mathcal{LTL}$. Some open remaining problems are:

1. Is there a finite basis for rules admissible in $\mathcal{LTL}$?
2. If there is no finite basis, whether an independent basis exists?
3. Is the unification problem in $\mathcal{LTL}$ for formulas with parameters (meta-variables) decidable?

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